UDC 539.3

ON THE ASYMPTOTIC VALUE OF THE UPPER CRITICAL PRESSURE OF NONSHALLOW SPHERICAL SHELLS

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The upper critical pressures of nonshallow spherical shells under uniform external pressure are determined by an asymptotic method [1, 2] as a function of the apperture angle and the mode of support. The case is considered when the parameter of relative thin-walledness is sufficiently small. The values found agree well with the results of computations by direct numerical methods and permit their continuation into the domain of arbitrarily thin shells where a machine computation becomes of very low efficiency.

1. Equations and boundary conditions. The Reissner equations for nonshallow shells under axisymmetric deformation [1] are considered:

$$\varepsilon^2\left\{\sin\xi\,(\Phi-\xi)''+\cos\xi\,(\Phi-\xi)'-\frac{\cos\Phi}{\sin\xi}(\sin\Phi-\sin\xi)+\nu(\cos\Phi-\cos\xi)\right\}= \tag{1.1}$$
$$\Psi\sin\Phi-T\cos\Phi$$
$$\varepsilon^2\left\{\sin\xi\Psi''+\cos\xi\Psi'-\left[\frac{\cos^2\Phi}{\sin\xi}-\nu\Phi'\cos\Phi\right]\Psi\right\}=\cos\Phi-\cos\xi+$$
$$\varepsilon^2\left\{\nu\sin\Phi T'-(\sin^2\xi p)'+\left[\frac{\sin\Phi\cos\Phi}{\sin\xi}+\nu\Phi'\cos\Phi\right]T-\nu\sin\xi p\cos\Phi\right\}$$

with each of the boundary conditions

$$\begin{array}{llll}
1)\ \Phi(0)=\Psi(0)=0, & M(b)=0, & \Psi(b)=-\varepsilon k u(b)\sin b & \quad (1.2)\\
2)\ \Phi(0)=\Psi(0)=0 & \Phi(b)=b, & N(b)=0 & \\
3)\ \Phi(0)=\Psi(0)=0, & \Phi(b)=b, & u(b)=0 &
\end{array}$$

All the quantities in (1.1), (1.2) are dimensionless and related to the dimensional quantities by the relationships

$$r_0=R\sin\xi,\quad z_0=R\cos\xi,\quad \gamma^2=12(1-\nu^2),\quad k_0=kE\gamma\varepsilon^3,\quad \varepsilon^2=\frac{h}{R\gamma}$$
$$p=\frac{p_H}{\varepsilon^4\gamma E},\quad q=\frac{p_V}{\varepsilon^4\gamma E},\quad p=-Q\sin\Phi,\quad q=Q\cos\Phi,\quad \Psi=\frac{\Psi_H}{\varepsilon^4R^2\gamma E}$$
$$T=\frac{\Psi_V}{\varepsilon^4R^2\gamma E}=-\int_0^\xi\sin\xi q\,d\xi,\quad N=\frac{N_\xi}{\varepsilon^4R\gamma E}=\frac{\Psi\cos\Phi+T\sin\Phi}{\sin\xi}$$
$$u=\frac{u_0}{R\varepsilon^2}=\sin\xi\Psi'+\sin^2\xi p-\nu\Psi\cos\Phi-\nu T\sin\Phi$$
$$M=\frac{M_\xi}{Eh^2\varepsilon^2}=\Phi'-1+\nu\frac{\sin\Phi-\sin\xi}{\sin\xi}$$

Here $\Phi(\xi)$ is an angle which the shell element makes with the horizontal axis before and after deformation at a point corresponding to the parameter ξ; Ψ_H, Ψ_V, p_H, p_V are the horizontal and vertical components of the stresses and loads, respectively, u_0 is the horizontal displacement, M_ξ is the bending moment, N_ξ is the radial force, E is Young's modulus, ν is the Poisson's ratio, h is the shell thickness, R is the shell radius, k_0 is the coefficient of elastic slip of the shell edge, and Q is the external load intensity.

The boundary conditions in (1.2) correspond to diverse methods of clamping the shell edge. Let us hence investigate the problem (1.1), (1.2) as $\varepsilon \to 0$.

2. Construction of the asymptotic expansions. The method of constructing the asymptotic expansions for the problem (1.1), (1.2) as $\varepsilon \to 0$ has been stated in [1]. Here, we limit ourselves to the construction of the principal terms of the asymptotics. Assuming $\varepsilon = 0$ in (1.1), we obtain

$$\Psi_0(\xi)\sin\Phi_0(\xi) + \cos\Phi_0(\xi)\int_0^\xi \sin\xi q(\xi, \Phi_0(\xi))\,d\xi = 0, \quad \cos\Phi_0(\xi) = \cos\xi \tag{2.1}$$

This system has the solution

$$\Phi_0(\xi) = \xi, \qquad \Psi_0(\xi) = -{}^1/_4\, Q \sin 2\xi \tag{2.2}$$

which corresponds to the membrane stressed equilibrium mode which agrees with the initial surface. The solution (2.2) satisfies (1.1) but does not satisfy the boundary conditions (1.2). Hence, the asymptotic representation of the solution corresponding to the equilibrium mode in the subcritical stage is expanded as $\varepsilon \to 0$ in the form

$$\Phi(\xi, \varepsilon) \sim \xi + g_0\left(\frac{b-\xi}{\varepsilon}\right), \quad \Psi(\xi, \varepsilon) \sim -\frac{1}{4} Q \sin 2\xi + h_0\left(\frac{b-\xi}{\varepsilon}\right) \tag{2.3}$$

Here the functions g_0, h_0 are concentrated in the edge effect zone (in the neighborhood of $\xi = b$) and cancel the residual in the functions $\Phi_0(\xi)$ and $\Psi_0(\xi)$ in complying with the boundary conditions (1.2). Furthermore, by using (2.3) we obtain for the right side of the first equation in (1.1), by taking account of (2.1) and (2.2),

$$I(\xi, \varepsilon) = (\Psi_0 + h_0)\sin(\xi + g_0) + Q\cos(\xi + g_0)\int_0^\xi \sin\xi\,[\cos(\xi + g_0) - \cos\xi]\,d\xi +$$

$$\frac{1}{2} Q \sin^2\xi \cos(\xi + g_0) = -\frac{1}{2} Q \sin\xi \sin g_0 + h_0 \sin(\xi + g_0) -$$

$$2Q\cos(\xi + g_0)\int_0^\xi \sin\xi \sin\left(\xi + \frac{g_0}{2}\right)\sin\frac{g_0}{2}\,d\xi$$

Let us substitute $\xi = b - \varepsilon t$ and let us find the principal term of the expansion in powers of εt in the neighborhood of the point $\xi = b$. Let us note that the integral in the last expression is a quantity of the order of ε, and equals

$$\varepsilon\int_b^\infty \sin b \sin\left[b + \frac{1}{2} g_0(t)\right]\sin\frac{g_0(t)}{2}\,dt + O(\varepsilon^2)$$

The solution (2.3) should go over into the solution (2.2) with distance from the boundary, hence $\{g_0(t), h_0(t)\} \to 0$ as $t \to \infty$. Let us assume that the last integral hence converges. Therefore, we have in the neighborhood $\xi = b$ as $\varepsilon \to 0$ the following equation:

$$I(\xi, \varepsilon) = -{}^1/_2\, Q \sin b \sin g_0(t) + h_0(t)\sin[b + g_0(t)] + O(\varepsilon)$$

Proceeding in an analogous manner with the remaining terms in (1.1), we arrive at a system of nonlinear differential equations with appropriate boundary conditions to determine the functions $g_0(t)$ and $h_0(t)$

$$\left.\begin{aligned}&\sin b\, g_0'' + {}^1/_2\, Q \sin b \sin g_0 - h_0 \sin(b + g_0) = 0\\ &\sin b\, h_0'' - \cos(b + g_0) + \cos b = 0\end{aligned}\right\} \qquad (2.4)$$

$$1)\ \ g_0'(0) = 0,\ \ h_0(0) + \Psi_0(b) = k \ \sin^2 b\, h_0'(0),\ \ g_0(\infty) = h_0(\infty) = 0 \qquad (2.5)$$

$$2)\ \ g_0(0) = 0,\ h_0(0) \cos b - {}^1/_2\, Q \sin b = 0,\ \ g_0(\infty) = h_0(\infty) = 0$$

$$3)\ \ g_0(0) = 0,\ h_0'(0) = 0,\ \ g_0(\infty) = h_0(\infty)$$

The conditions at infinity are obtained from the requirement that the asymptotic solution (2.3) goes over into (2.2) within the domain.

In the case of the boundary conditions (3) in (2.5) (an absolutely fixed support) and (1) in (2.5) for $k = \infty$ (fixed hinge), it can be shown that, following [2], the values of the upper critical pressure $Q^* = 4$. Hence, $g_0 = h_0 = 0$, and the asymptotic representations become

$$\Phi(\xi, \varepsilon) \sim \xi + \varepsilon g_1\left(\frac{b - \xi}{\varepsilon}\right), \qquad \Psi(\xi, \varepsilon) \sim -\frac{1}{4} Q \sin 2\xi + \varepsilon h_1\left(\frac{b - \xi}{\varepsilon}\right)$$

The functions g_1, h_1 are here determined from the equations and boundary conditions

$$g_1'' + {}^1/_2\, Q g_1 - h_1 = 0, \qquad h_1'' + g_1 = 0$$

$$1)\ \ g_1(0) = 0,\ \ h_1'(0) = -{}^1/_2\, Q(1 - \nu), \qquad g_1(\infty) = h_1(\infty) = 0$$

$$2)\ \ g_1'(0) = 0,\ \ h_1'(0) = -{}^1/_2\, Q(1 - \nu), \quad g_1(\infty) = h_1(\infty) = 0$$

The formulas for g_1, h_1 are easily written down explicitly.

3. Solution of the edge effect equations (2.4), (1), (2) **in** (2.5). Let us seek the least value of the parameter Q for which still one solution is manifest in any sufficiently small neighborhood in the problem (2.4), (1), (2) in (2.5).

To do this, let us use a method similar to that stated in [2 – 4]. We seek the solution in the form

$$g_0 = \sum_{m+n\geqslant 1}^{N} g_{mn} z_1^m z_2^n, \qquad h_0 = \sum_{m+n\geqslant 1}^{N} h_{mn} z_1^m z_2^n \qquad (3.1)$$

$$z_1 = c_1 e^{r_1 t},\ z_2 = c_2 e^{r_2 t},\ r_1 = -a - ib,\ r_2 = -a + ib$$

$$a = ({}^1/_2 - {}^1/_8\, Q)^{1/2}, \qquad b = ({}^1/_2 + {}^1/_8\, Q)^{1/2}$$

Here z_1, z_2 are fundamental solutions of the linearized system (2.4) which decrease at infinity

$$g_0'' + {}^1/_2\, Q g_0 - h_0 = 0, \qquad h_0'' + g_0 = 0$$

Extracting the linear terms, let us rewrite (2.4) in the form

$$\begin{aligned}&g_0'' + {}^1/_2\, Q g_0 - h_0 = h_0(\cos g_0 - 1) - {}^1/_2\, Q(\sin g_0 - g_0) + \operatorname{ctg} b h_0 \sin g_0\\ &h_0'' + g_0 = \operatorname{ctg} b(\cos g_0 - 1) - (\sin g_0 - g_0)\end{aligned} \qquad (3.2)$$

Now, substituting (3.1) into (3.2) and equating terms of identical powers of $z_1^m z_2^n$, we obtain a recurrent infinite system of linear algebraic equations to determine the complex coefficients g_{mn}, h_{mn}

$$\begin{aligned}&g_{mn} + (mr_1 + nr_2)^2 h_{mn} = G_1(g_{kp}, h_{kp})\\ &[(mr_1 + nr_2)^2 + {}^1/_2\, Q]\, g_{mn} - h_{mn} = G_2(g_{kp}, h_{kp})\end{aligned} \qquad (3.3)$$

Here $G_1(g_{kp}, h_{kp})$, $G_2(g_{kp}, h_{kp})$ are functions corresponding to the right sides of the system (3.2) which depend on g_{kp}, h_{kp}, where $k + p < m + n$ $(G_1 \equiv G_2 \equiv 0$ for

$m+n=1$). Let us limit ourselves to a finite number of terms N in (3.1). We hence replace $\sin g_0$, $\cos g_0$ in the right side of (3.2) by a series expansion in g_0, and we select the number of terms as a function of N.

Since an ambiguity in the determination of h_{10}, h_{01} exists in (3.3), for convenience we shall take $h_{10}=\overline{h_{01}}$. Hence, for any $m+n\geqslant 1$ we obtain $g_{mn}=\overline{g_{nm}}$, $h_{mn}=\overline{h_{nm}}$. Therefore, all the coefficients g_{nm}, h_{mn} can be calculated by means of h_{10}, h_{01}.

To determine c_1, c_2 let us substitute (3.1) with known coefficients into the boundary conditions (2.5). We obtain a system of two nonlinear equations

$$1)\ \Phi_1(c_1,c_2,Q)\equiv\sum_{m+n\geqslant 1}^{N}(mr_1+nr_2)\,g_{mn}c_1^{\,m}c_2^{\,n}=0 \qquad (3.4)$$

$$\Phi_2(c_1,c_2,Q)\equiv-\sum_{m+n\geqslant 1}^{N}[1-k\sin^2 b\,(mr_1+hr_2)]\,h_{mn}c_1^{\,m}c_2^{\,n}+\frac{1}{4}Q\sin 2b=0$$

$$2)\ \Phi_1(c_1,c_2,Q)\equiv\sum_{m+n\geqslant 1}^{N}g_{mn}c_1^{\,m}c_2^{\,n}=0$$

$$\Phi_2(c_1,c_2,Q)\equiv-\sum_{m+n\geqslant 1}^{N}\cos b\,h_{mn}c_1^{\,m}c_2^{\,n}+\frac{1}{2}Q\sin b=0$$

The counting process starts for the value of the parameter $Q=0$, for which the problem (2.4), (2.5) has the trivial solution $c_1=c_2=0$. The values $c_1(Q)$, $c_2(Q)$ are sought by the Newton method in combination with the method of sequential loading by the parameter Q [2].

Appropriate computations by means of the algorithm described were carried out on an "Odra-1024" computer. The selection of the spacing during motion in the parameter was accomplished automatically, and the terms in identical powers of $z_1^{\,m}z_2^{\,n}$ in the right side were grouped by using a polynomial working program. The computations were checked by using the first integral

$$\tfrac{1}{2}\sin b\,[(g_0')^2-(h_0')^2]-\tfrac{1}{2}Q\sin b\,(\cos g_0-1)-h_0[\cos(b+g_0)-\cos b]=0$$

which is obtained from (2.4) if the first equation is multiplied by $-g_0'$, the second by h_0' and added and integrated between t and ∞. Setting $t=0$, and using the appropriate boundary conditions, we obtain verification formulas for each case of support. Results of computing Q^* for the boundary conditions (1) from (1.2) are presented below for $k=0$ (lower line) and (2) from (1.2) (upper line).

$b=0.2$,	0.4,	0.6,	0.8,	1.0,	1.2,	1.4,	1.57
$Q^*=1.753$,	1.721,	1.703,	1.557,	1.596,	1.505,		
$Q^*=0.796$	0.815,	0.858,	0.932,	1.046,	1.209,	1.472,	2

The dependence $Q^*(k)$ for fixed angles b (boundary conditions (1) from (1.2) for $k\neq 0$) is shown in Fig. 1.

Let us note that (2.3), (2.4) are invariant relative to replacement of b by $\pi-b$, g_0 by $-g_0$, h_0 by $-h_0$. Hence, the results for $b>\pi/2$ can be obtained from those presented by replacing b by $\pi-b$.

4. Asymptotic value of the upper critical load. Let us turn to the dimensional variable $P=QE\gamma\varepsilon^4$. Then for sufficiently thin shells under the support

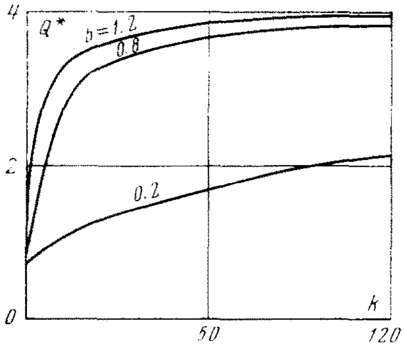

Fig. 1

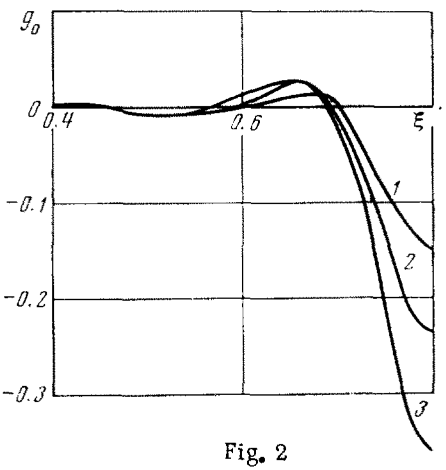

Fig. 2

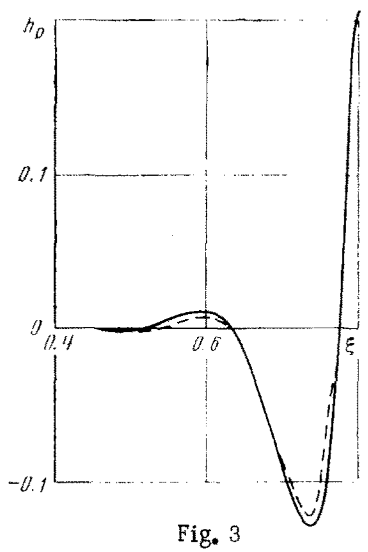

Fig. 3

conditions (1), (2) from (1.2), the values of the upper critical loads are determined by the formula

$$P = \frac{Q^* E}{\sqrt{12(1-\nu^2)}} \left(\frac{h}{R}\right)^2 \left[1 + a_1\left(\frac{h}{R}\right) + \ldots\right]$$

where Q^* have been found in Sect. 3. The quantities a_i were not sought here. In the case of an absolutely fixed support and a fixed hinge (boundary condition (3) from (1.2) for $k = \infty$), $Q^* = 4$. This result has been obtained earlier by Pogorelov [5].

Shown in Fig. 2 is the development of the edge effect as a function of the magnitude of the load Q (boundary condition (1) from (1.2) for $k = 0$, $b = 0.8$ for $\varepsilon^2 = 0.109 \cdot 10^{-2}$). The values of Q equal to 0.458, 0.639, 0.827 correspond to curves 1 – 3.

It is seen that buckling, i.e. the appearance of new equilibrium modes, is determined mainly in the edge effect zone. Shown for comparison in Fig. 3 are the results of computing $h_0(\xi)$ by the method of "adjustment" [6] (solid line) and the asymptotic method (dashes) for $Q = 0.827$, $\varepsilon^2 = 0.109 \cdot 10^{-2}$ for $\xi \in [0.4,\ 0.8]$ (for $\xi < 0.4$, $|h_0(\xi)| \leqslant 10^{-4}$).

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PROBLEM OF THE OPTIMUM SKI JUMP

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Depending on the properties of the ski jump, a skier completes his flight in 2.5–4.5 sec and can influence his motion trajectory in the air by changing the angle of attack of the body. Query: How should a sportsman control his body in flight in order to touch down at the greatest distance?

A formulation of this problem and its numerical solution on an electronic computer as an optimum control problem are presented below under the following assumptions: motion of the center of mass of a skier – ski system subjected to gravity, a drag R and a lift Y is considered. The equations of motion and the initial conditions are

$$\frac{dv}{dt} = -\frac{R}{m} - g\sin\theta, \qquad \frac{d\theta}{dt} = \frac{Y}{mv} - \frac{g\cos\theta}{v}$$
$$\frac{dx}{dt} = v\cos\theta, \qquad \frac{dy}{dt} = v\sin\theta$$
$$R = {}^1/_2\,\rho\, v^2 S c_x, \qquad Y = {}^1/_2\,\rho\, v^2 S c_y$$
$$t = 0,\ x = 0,\ y = 0,\ v = v_0,\ \theta = \theta_0$$

Here t is the time, x, y, v, θ are the horizontal range, height, modulus of the velocity and slope of the velocity to the x-axis, respectively, m is the system mass, g the acceleration of gravity, ρ the air density, S the characteristic area, and c_x and c_y the aerodynamic coefficients.

The dependence of c_y and c_x on the angle of attack α are taken from [1], where experimental curves obtained as a result of wind tunnel tests on skiers are presented. For convenience in the calculations, these curves have been approximated by the dependencies $c_y(\alpha) = -0.000250\,\alpha^2 + 0.0228\,\alpha - 0.0920$ and $c_x(\alpha) = 0.0103\,\alpha$. The angle of attack can vary between $\alpha_{\min}(t)$ and $\alpha_{\max}(t)$.

The motion is modeled on the profile of the Planitsa (Yugoslavia) jump, on which the absolute world's record jump of 165 m was established (see Fig. 1; a diagram of the angles and forces acting on the skier in flight is given in the upper right corner).